Optimal Risk Control under Excess of Loss Reinsurance

Mohamed Mnif, Agnès Sulem

To cite this version:

Mohamed Mnif, Agnès Sulem. Optimal Risk Control under Excess of Loss Reinsurance. [Research Report] RR-4317, INRIA. 2001. inria-00072270

HAL Id: inria-00072270

https://hal.inria.fr/inria-00072270

Submitted on 23 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Optimal risk control under excess of loss reinsurance

Mohamed Mnif and Agnès Sulem

N° 4317

November, 14 2001

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--4317--FR+ENG

Optimal risk control under excess of loss reinsurance

Mohamed Mnif * and Agnès Sulem†

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet MATHFI

Rapport de recherche n° 4317 — November, 14 2001 — 28 pages

Abstract: We study the optimal reinsurance policy of an insurance company which gives part of its premium stream to another compagny in exchange of an obligation to support the difference between the amount of the claim and some retention level. This contract is known as excess of loss reinsurance. The objective of the insurance compagny is to maximize the expected utility of its reserve at some planning horizon and under a nonnegativity constraint. We suppose that reinsurance incurs a cost proportional to the size of risk run by the reinsurance compagny. We first prove existence and uniqueness results for this optimization problem by using stochastic control methods. In a second part, we solve the associated Bellman equation numerically by using an algorithm based on policy iterations.

Key-words: Stochastic control, state constraint, dynamic programming principle, viscosity solution, Howard algorithm, insurance, reinsurance, risk control.

* INRIA-Rocquencourt et Laboratoire de Probabilités et Modèles Aléatoires, CNRS, UMR 7599, UFR Mathématiques, Case 7012, Université Paris 7, 2 Place Jussieu 75251 Paris Cedex 05, e-mail: mnif@gauss.math.jussieu.fr

† INRIA-Rocquencourt, email: agnes.sulem@inria.fr

Contrôle de risque avec réassurance d'excès de pertes

Résumé : On étudie la politique optimale de réassurance d'une compagnie d'assurance qui reverse une partie des primes qu'elle reçoit à une autre compagnie en échange d'un engagement de celle-ci à payer la différence entre la taille de chaque sinistre qui survient et un certain niveau de rétention donné.Ce contrat est connu sous le nom de réassurance d'excès de pertes. L'objectif de la compagnie d'assurance est de maximiser l'espérance de l'utilité de son capital terminal sous une contrainte de positivité du capital à tout instant. On suppose que la réassurance engendre un coût proportionel à la taille du risque encouru par la compagnie de réassurance. On prouve l'existence et l'unicité de la solution de ce problème d'optimisation en utilisant des méthodes de contrôle stochastique. Dans une seconde partie, on résoud numériquement l'équation de Bellman associée à ce problème en utilisant un algorithme basé sur l'algorithme d'Howard.

Mots-clés : Contrôle stochastique, Contrainte d'état, principe de programmation dynamique, solution de viscosité, algorithme de Howard, assurance, réassurance, contrôle de risque.

1 Introduction

This paper concerns the theoretical and numerical study of optimal risk control of an insurance company. The reserve process of the insurance compagny consists of a premium stream which commits the compagny to pay the amount of the claims at their arrival.
To reduce the risk, the insurance compagny gives part of the premium stream to another compagny, in exchange of an obligation to support the difference between the amount of each claim and some fixed level called retention level. This contract is known as excess of loss reinsurance. We suppose that reinsurance incurs a cost proportional to the size of the risk taken by the reinsurance compagny.
The objective of the insurance compagny is to maximize the expected utility of the terminal wealth over all admissible strategies which satisfies a nonnegative wealth constraint over the whole interval $[0, T]$, and to determine the optimal policy of reinsurance. This problem was formulated by Asmussen, Højgaard and Taksar (2000) who considered the issue of optimal risk control and dividend distribution policies. They modelled the wealth process as a diffusion process and reparametrized the problem by considering the drift term of the diffusion as the basic control parameter, which leads to a standard stochastic control problem.
In this paper, we model the risk process of the insurance compagny by using a compound Poisson process. In Mnif and Pham (2001), stochastic optimization problem is studied when the state process belongs to a convex family of semimartingales. Existence and uniqueness of the solution of the optimization problem is proved by converting the dynamic problem into a static problem. The characterization of the solution is obtained by using a dual formulation. In the present paper, the convexity property of the risk process does not hold because of the excess of loss contract, in contrast with the proportional reinsurance case (see Touzi (2000)). Due to the Markovian context, the reinsurance problem may be studied by a direct dynamic programming Hamilton Jacobi Bellman equation (HJB in short).
For control problems and associated HJB equations, the notion of viscosity solution, first introduced by Crandall and Lions in 1983, is known to be a powerful tool. Here, owing to the state constraint, we need to consider contrained viscosity solutions. This notion was first introduced by Soner (1986a).
The purpose of this paper is to prove an existence and uniqueness result for the dynamic programming equation associated to this problem and then to solve it by using an efficient numerical method, the convergence of which is ensured by the uniqueness result.
The paper is organised as follows. The problem is formulated in Section 2. In Section 3, we study the properties of the value function. In Section 4, we prove that the value function is a constrained viscosity solution of the associated HJB equation. In Section 5 we prove the uniqueness of the solution of the HJB. Section 6 is devoted to numerical analysis of the HJB equation: The HJB equation is discretized by using finite difference schemes and solved by

using an algoritm based on the “Howard algorithm”(policy iteration). Numerical results are presented. They provide the optimal policy of reinsurance.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We consider an integer valued random measure $\mu(dt, dz)$, defined on $(\Omega, \mathcal{F}, P)$, associated to the marked point process $(N, \{Y(n), n \in \mathbb{N}\})$. Here, $\{N(t), t \geq 0\}$ is a counting process corresponding to the random time points $\{T_n, n \in \mathbb{N}\}$ of arrival of the claims and $\{Y(n), n \in \mathbb{N}\}$ is a sequence of random variables with values in the mark space $B \subset \mathbb{R}_+$. We take $B = [b, \infty)$ with $b > 0$. As usual, μ and $(N, \{Y(n), n \in \mathbb{N}\})$ are identified by the formula

$$\mu([0,t] \times A) = \sum_{n \geq 1} 1_{\{T_n \leq t\}} 1_A(Y_n),$$

for all $t \in [0, T]$ and $A \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-field on B.
We assume that μ is right-continuous and $\mu(0, A) = 0$ for all $A \in \mathcal{B}$. We denote by $\mathbb{F} = \{\mathcal{F}(t), 0 \leq t \leq T\}$ the P-completed filtration generated by the random measure $\mu(dt, dz)$. We assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. The random measure $\mu(dt, dz)$ is assumed to have an intensity measure $q(dt, dz) = \pi(dz)dt$ with $\int_B \pi(dz) < \infty$ which means that there is a finite number of jumps during any finite time interval. By definition of the intensity $q(dt, dz)$, the compensated jump process defined by:

$$\tilde{\mu}(dt, dz) = \mu(dt, dz) - \pi(dz)dt,$$

is such that $\{\tilde{\mu}([0,t] \times A), 0 \leq t \leq T\}$ is a $(P, \mathcal{F}(t))$ martingale for all $t \in [0, T]$ and $A \in \mathcal{B}$. We shall assume that the following non degeneracy condition holds:

$$\underline{\pi} \leq \int_B \pi(dz) < \infty, \tag{2.1}$$

for some $\underline{\pi} > 0$, and

$$\int_B z^2 \pi(dz) < \infty. \tag{2.2}$$

Example 2.1 *Suppose that π is a finite measure on B such that $\pi(dz) = \lambda h(z)dz$, where h is a probability density which admits second order moment. Then the technical conditions given above are satisfied.*

A retention level process is an $(\mathcal{F}_t)$-adapted process $\alpha = (\alpha_t, 0 \leq t \leq T)$ interpreted as the risk level which is supported by the insurance compagny.

Given a retention level α, we denote by $p(\alpha_t)$ the difference between the premium rate per unit of time received by the compagny and the premium rate per unit of time paid by the compagny to the reinsurer.
The risk process of the insurance company under this excess of loss contract is then given by :

$$X_s^{t,x,\alpha} \;=\; x + \int_t^s p(\alpha_u)du - \int_t^s \int_B z \wedge \alpha_u \, \mu(du, dz), \;\; t \leq s \leq T,$$

for an initial reserve x at time t.
Observe that the state process can also be expressed as:

$$X_s^{t,x,\alpha} \;=\; x + \int_t^s p(\alpha_u)du - \sum_{i=1}^{N_s} U_i \wedge \alpha,$$

where N_s is a Poisson process with intensity $\beta = \int_B \pi(dz)$, and U_i are i.i.d random variables independant of N_s, having finite first moment ν.
>From now on, we consider a premium rate of the same form as in Asmussen, Højgaard and Taksar (2000):

$$p(\alpha) = \beta\nu - (1 + k(\alpha))\beta(\nu - E[U_i \wedge \alpha]), \tag{2.3}$$

where $k(\alpha)$ is a proportional factor satisfying $k(0) = 0$ and $k(\alpha) = k > 0$ if $\alpha > 0$.
The first term of the r.h.s of Equation (2.3) is the expectation of the amount of the claims during a unit of time. The second term of the r.h.s of Equation (2.3) is the premium, which is given to another compagny of insurance to support the difference between the amount of the claims and the retention level α during a unit of time.
We impose the following constraint on the family of state process X:

$$X_s^{t,x,\alpha} \geq 0, \;\; \text{for all } t \leq s \leq T. \tag{2.4}$$

Given $K > 0$, we impose

$$\alpha_s \leq K \;\; \text{for all } t \leq s \leq T. \tag{2.5}$$

An admissible policy α is an adapted stochastic process $(\alpha_s)_{t \leq s \leq T}$, right continuous, such that Conditions (2.4) and (2.5) are satisfied. We denote by $\mathcal{A}(t, x)$ the set of all admissible policies.

Remark 2.1 *The set* $\mathcal{A}(t, x)$ *is not empty, since* $0 \in \mathcal{A}(t, x)$ *for all* $t \in [0, T]$ *and* $x \in I\!R_+$.

Let U denote a utility function such that

$$U \text{ is uniformly continuous, non decreasing and concave }. \tag{2.6}$$

The objective of the insurance campagny is to maximize the expectation of the utility of the terminal value of the risk process defined by

$$J(t, x, \alpha) := E(U(X_T^{t,x,\alpha})),$$

over all policies $\alpha \in \mathcal{A}(t, x)$. We define the value function v as

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(t,x)} J(t, x, \alpha). \tag{2.7}$$

The terminal condition is given by

$$v(T, x) = U(x). \tag{2.8}$$

Remark 2.2 *When the state process reaches zero, it remains there. Because of the jump term and the state constraint, the only admissible retention level is* $\alpha = 0$. *Since* $p(0) = 0$, *we have* $v(t, 0) = 0$ *for all* $t \in [0, T]$.

3 Properties of the value function

In this section, we focus on proving that the value function v belongs to $UC_x([0,T] \times I\!R_+)$, the set of continuous functions in $[0,T] \times I\!R_+$ and uniformly continuous in x.
By the dynamic programming principle, we have :

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(t,x)} E(v(t + h \wedge \tau, X_{t+h\wedge\tau,}^{t,x,\alpha})), \tag{3.1}$$

for any stopping time $\tau \in \mathcal{T}_{t,T}$, $0 \leq t \leq T$ and $0 < h < T - t$, where $\mathcal{T}_{t,T}$ is the set of all stopping times between t and T.
Let us establish some preliminary estimates on the moments of the state process.

Lemma 3.1 *For any* $k \in [0, 2]$, *there exists* $C' = C'(k, T) > 0$, *such that for all* $t \in [0, T]$, $x \in I\!R_+$ *and* $0 < h < T - t$:

$$E \mid X_{t+h}^{t,x,\alpha} - x \mid^k \leq C' h^{\frac{k}{2}}. \tag{3.2}$$

Proof. According to the Hölder inequality, it suffices to prove (3.2) for $k = 2$. For notational simplicity, hereafter, C denotes a generic constant. Applying Itô's lemma to $(X_{t+h}^{t,x,\alpha} - x)^2$, we obtain

$$E \mid X_{t+h}^{t,x,\alpha} - x \mid^2 \leq C' \left(E \left[\int_t^{t+h} p(\alpha_s)^2 ds \right] + E \left[\int_t^{t+h} \int_B (z \wedge \alpha_s)^2 m(dz) ds \right] \right).$$

Since $p(\alpha_s)$ is bounded and $z \wedge \alpha_s \leq z$ for all $s \in [t,T]$ and for all $z \in B$, by (2.2) we have:

$$E \mid X_{t+h}^{t,x,\alpha} - x \mid^2 \leq C'h,$$

which implies (3.2). □

In the following lemma, we prove that when the reserve process reaches a low level at time $s \in [t,T]$, the optimal control is $\alpha_u = 0$ for all $u \in [s,T]$.

Lemma 3.2 *There exists $\underline{x} \in I\!\!R_+$, such that for all $t \in [0,T]$ and $x \in I\!\!R_+$, the following property holds: if there exist $s \in [t,T]$ and $\alpha \in A(t,x)$ such that $X_s^{t,x,\alpha} < \underline{x}$, then the optimal control is $\alpha_u = 0$ for all $u \in [s,T]$.*

Proof. >From the premium rate expression (2.3), we have

$$p(\alpha) = \begin{cases} -k\beta\nu + (1+k)\nu E[U_i \wedge \alpha] & \text{if } \alpha > 0 \\ 0 \quad \text{if } \alpha = 0. \end{cases}$$

Let $\underline{x} := \frac{k\beta}{1+k}$. In this case :

$$\text{if } 0 < \alpha < \underline{x} \text{ then } p(\alpha) < 0. \tag{3.3}$$

Define $u_1 := \inf\{u \geq s \text{ such that } \alpha_u \geq \underline{x}\}$. Since the control α is rightcontinuous,

$$\alpha_{u_1} \geq \underline{x}. \tag{3.4}$$

Using the hypothesis $X_s^{t,x,\alpha} < \underline{x}$, the definition of u_1 and (2.3), we obtain

$$\begin{aligned} 0 &\leq X_{u_1}^{t,x,\alpha} \\ &= X_s^{t,x,\alpha} + \int_s^{u_1} p(\alpha_t)dt - \int_s^{u_1}\int_B z \wedge \alpha_t \, \mu(dt,dz) \\ &< \underline{x} - \int_s^{u_1}\int_B z \wedge \alpha_t \, \mu(dt,dz), \end{aligned}$$

which implies that

$$\int_s^{u_1}\int_B z \wedge \alpha_t \, \mu(dt,dz) < \underline{x},$$

and consequently

$$z \wedge \alpha_{u_1} < \underline{x} \quad \text{for all } z \in B.$$

For z large enough, we have: $\alpha_{u_1} < \underline{x}$, which contradicts Inequality (3.4). Consequently all admissible controls must satisfy: for all $u \geq s$, $\alpha_u < \underline{x}$. Using (2.3) and since $p(\alpha) = 0$, if

$\alpha = 0$, the optimal control is $\alpha_u = 0$ for all $u \in [s,T]$. □

Define $A^{\underline{x}}(t,x)$ as:

$$\begin{aligned} A^{\underline{x}}(t,x) \ &:= \ \{\alpha \in A(t,x) \text{ such that: if there exists } \ s \in [t,T] \text{ such that } X_s^{t,x,\alpha} \leq \underline{x}, \\ &\quad \text{then for all } u \in [s,T]\ , \alpha_u = 0\}. \end{aligned}$$

Lemma 3.3 *Let x, y be such that $0 < x - y < \underline{x}$. Then,*

$$A^{\underline{x}}(t,x) \subset A(t,y) \subset A(t,x).$$

Proof. The second inclusion is obvious since $x > y$. It remains to prove the first inclusion. Let $\alpha \in A^{\underline{x}}(t,x)$. Define

$$\tau = \inf\{s \geq t \text{ such that } X_s^{t,x,\alpha} < \underline{x}\} \wedge T.$$

If $s < \tau$, then $X_s^{t,x,\alpha} \geq \underline{x}$ which implies $X_s^{t,y,\alpha} \geq 0$. If $s \geq \tau$, then $X_s^{t,x,\alpha} < \underline{x}$ and so $\alpha_s = 0$, which proves that $\alpha \in A(t,y)$. □

Let us approximate the function U by a sequence of Lipschitz functions.

Lemma 3.4 *Under Assumptions (2.6), there exists a sequence of Lipschitz functions $(U^n, n \geq 0)$ which converges uniformly to U.*

Proof. Since U is uniformly continuous in x, we can define the modulus of continuity

$$w_u(r) = \sup\{U(x) - U(y) \text{ such that } \mid x - y \mid \leq r \text{ and } x, y \in I\!\!R_+\}.$$

Fix $n \in I\!\!N$. Define U^n as follows:

$$U^n(x) \ := \ \begin{cases} U(x), \ \text{ if } \ x = \dfrac{p}{n} \ \text{ where } \ p \in I\!\!N, \\ n(U(\dfrac{p+1}{n}) - U(\dfrac{p}{n}))(x - \dfrac{p}{n}) + U(\dfrac{p}{n}) \ \text{ if } \ \dfrac{p}{n} < x \leq \dfrac{p+1}{n}. \end{cases}$$

For all $x \in I\!\!R_+$, there exists $p \in I\!\!N$ such that $\frac{p}{n} \leq x \leq \frac{p+1}{n}$. Using the concavity and the monotonicity of U, we have

$$0 \leq U(x) - U^n(x) \leq U(\frac{p+1}{n}) - U(\frac{p}{n}),$$

and thus

$$\sup_{x \in I\!\!R_+} \mid U(x) - U^n(x) \mid \leq w_u(\frac{1}{n}).$$

This implies the uniform convergence of U^n to U, when n goes to ∞. It remains to prove the Lipschitz property of the function U^n. Let $x, y \in I\!\!R_+$. We suppose that $x > y$. There exists $k, l \in I\!\!N$ such that $x \in [\frac{k}{n}, \frac{k+1}{n})$ and $y \in [\frac{l}{n}, \frac{l+1}{n})$.

- Suppose that $k-1 \geq l+1$. Then,

$$\begin{aligned}
U^n(x) - U^n(y) &= U^n(x) - U^n(\frac{k}{n}) + \sum_{i=l+1}^{k-1} (U^n(\frac{i+1}{n}) - U^n(\frac{i}{n})) + U^n(\frac{l+1}{n}) - U^n(y) \\
&= n(x - \frac{k}{n})(U(\frac{k+1}{n}) - U(\frac{k}{n})) + \sum_{i=l+1}^{k-1} (U(\frac{i+1}{n}) - U(\frac{i}{n})) \\
&- n(y - \frac{l+1}{n})(U(\frac{l+1}{n}) - U(\frac{l}{n})) \\
&\leq nw_u(\frac{1}{n})(x-y) + \sum_{i=l+1}^{k-1} (U(\frac{i+1}{n}) - U(\frac{i}{n})) \\
&\leq 2nw_u(\frac{1}{n})(x-y).
\end{aligned}$$

- Suppose that $k-1 = l$, then

$$\begin{aligned}
U^n(x) - U^n(y) &= U^n(x) - U^n(\frac{k}{n}) + U^n(\frac{l+1}{n}) - U^n(y) \\
&\leq nw_u(\frac{1}{n})(x-y).
\end{aligned}$$

- Suppose that $k = l$, then

$$U^n(x) - U^n(y) \leq nw_u(\frac{1}{n})(x-y),$$

which proves the Lipschitz property of U_n. □

Consider now the auxiliar value function

$$v^n(t,x) := \sup_{\alpha \in A(t,x)} J^n(t,x,\alpha),$$

where $J^n(t,x,\alpha) := E[U^n(X_T^{t,x,\alpha})]$. In the following lemma, we study the properties of the function v^n.

Lemma 3.5 *Let $t \in [0,T]$ and $x \in \mathbb{R}_+$, then the function v^n is non decreasing Lipschitz in x and satisfies*

$$v^n(t,x) = \sup_{\alpha \in A^{\underline{x}}(t,x)} J^n(t,x,\alpha). \tag{3.5}$$

Proof. The monotonicity of the function v^n in x is trivial. Equality (3.5) is a consequence of Lemma (3.2) . To prove the Lipschitz property , we fix $x, y \in I\!R_+$ such that $x > y$. There exists a subdivision $y = z_0 < z_1 < ... < z_m = x$, where $m \in I\!N$ such that $0 < z_i - z_{i-1} \leq \underline{x}$ $\forall i \in [1, m]$.

$$\begin{aligned}
v^n(t,x) - v^n(t,y) &= \sum_{i=1}^{m} v^n(t,z_i) - v^n(t,z_{i-1}) \\
&= \sum_{i=1}^{m} \sup_{\alpha \in A(t,z_i)} E[U^n(X_T^{t,z_i,\alpha})] - \sup_{\alpha \in A(t,z_{i-1})} E[U^n(X_T^{t,z_{i-1},\alpha})] \\
&= \sum_{i=1}^{m} \sup_{\alpha \in A^{\underline{x}}(t,z_i)} E[U^n(X_T^{t,z_i,\alpha})] - \sup_{\alpha \in A(t,z_{i-1})} E[U^n(X_T^{t,z_{i-1},\alpha})] \\
&\leq \sum_{i=1}^{m} \sup_{\alpha \in A(t,z_{i-1})} E[U^n(X_T^{t,z_i,\alpha})] - \sup_{\alpha \in A(t,z_{i-1})} E[U^n(X_T^{t,z_{i-1},\alpha})] \\
&\leq \sum_{i=1}^{m} K^n \mid z_i - z_{i-1} \mid \\
&= K^n \mid x - y \mid,
\end{aligned}$$

where the first inequality is deduced from Lemma (3.3),the last inequality is deduced from Lemma (3.4) and K^n is the constant of Lipschitz of U^n. This proves the Lipschitz property in x of the function v^n. □

The regularity of the value function v can be stated as follows:

Theorem 3.1 *Under Assumptions (2.6), the function $v \in UC_x([0,T] \times I\!R_+)$.*

Proof. Let $t \in [0,T]$, $x \in I\!R_+$. Since $U^n(x) \leq U(x)$ for all $n \in I\!N$, we have

$$0 \leq v(t,x) - v^n(t,x) \leq \sup_{\alpha \in A(t,x)} E\left[U(X_T^{t,x,\alpha}) - U^n(X_T^{t,x,\alpha})\right].$$

Using the uniform convergence of U^n towards U in $I\!R_+$, we obtain that $v^n(t,x)$ converges to $v(t,x)$ when $n \longrightarrow \infty$ for all $x \in I\!R_+$, which proves the uniform convergence of $v^n(t,.)$ to $v(t,.)$ in $I\!R_+$. From Lemma (3.5), we deduce the uniform continuity of v in x. It remains to prove the continuity of v in $[0,T] \times I\!R_+$. Let $(x,y) \in I\!R_+^2$, $0 \leq t < s \leq T$. Using the dynamic programming principle (3.1) with $h = s - t$, Lemma (3.5) and Lemma (3.1) we obtain

$$\begin{aligned}
\mid v^n(s,y) - v^n(t,x) \mid &\leq \mid v^n(s,y) - v^n(t,y) \mid + \mid v^n(t,y) - v^n(t,x) \mid \\
&= \mid v^n(s,y) - \sup_{\alpha \in A(t,y)} E\left[v^n(s, X_s^{t,y,\alpha})\right] \mid + \mid v^n(t,y) - v^n(t,x) \mid \\
&\leq K^n \sup_{\alpha \in A(t,y)} E \mid X_s^{t,y,\alpha} - y \mid + K^n \mid x - y \mid \\
&\leq K^n C' \sqrt{\mid s - t \mid} + K^n \mid x - y \mid .
\end{aligned}$$

Since v^n converges uniformly to v, we conclude that v is continuous in $[0,T] \times I\!R_+$. □

4 Viscosity solution

As it is known, the dynamic programming principle yields that the value function is a viscosity solution of the corresponding HJB equation (see Fleming and Soner (1993)). Here, because of the state constraint, we will prove that the value function is a constrained viscosity solution of the following HJB equation

$$\max_{\alpha\in[0,K],\ x\geq z\wedge\alpha\ \forall z\in B}\left\{\frac{\partial\psi}{\partial t}(t,x)+A^{\alpha}(t,x,\psi(t,x),\frac{\partial\psi}{\partial x}(t,x))\right\}=0 \text{ in } [0,T]\times I\!R_+, \tag{4.1}$$

where

$$A^{\alpha}(t,x,\psi(t,x),\frac{\partial\psi}{\partial x}(t,x))=p(\alpha)\frac{\partial\psi}{\partial x}(t,x)+\int_B(\psi(t,x-z\wedge\alpha)-\psi(t,x))\pi(dz).$$

We first recall the definition of constrained viscosity solutions.

Definition 4.1 *(i)A function v in $C^0([0,T]\times I\!R_+)$ is a viscosity supersolution of (4.1) in $[0,T]\times I\!R_+$ if*

$$\max_{\alpha\in[0,K],\ x\geq z\wedge\alpha\ \forall z\in B}\left\{\frac{\partial\psi}{\partial t}(t,x)+A^{\alpha}(t,x,\psi(t,x),\frac{\partial\psi}{\partial x}(t,x))\right\}\leq 0,$$

whenever $\psi\in C^1([0,T]\times I\!R_+)$ and $v-\psi$ has a global minimum at $(t,x)\in[0,T]\times I\!R_+$.
(ii)A function v in $C^0([0,T]\times I\!R_+)$ is a viscosity subsolution of (4.1) in $[0,T]\times I\!R_+^*$ if

$$\max_{\alpha\in[0,K],\ x\geq z\wedge\alpha\ \forall z\in B}\left\{\frac{\partial\psi}{\partial t}(t,x)+A^{\alpha}(t,x,\psi(t,x),\frac{\partial\psi}{\partial x}(t,x)))\right\}\geq 0,$$

whenever $\psi\in C^1([0,T]\times I\!R_+^*)$ and $v-\psi$ has a global maximum at $(t,x)\in[0,T]\times I\!R_+^*$.
(iii)A function v in $C^0([0,T]\times I\!R_+)$ is a constrained viscosity solution of (4.1) in $[0,T]\times I\!R_+$ if it is both a viscosity supersolution of (4.1) in $[0,T]\times I\!R_+$ and a viscosity subsolution of (4.1) in $[0,T]\times I\!R_+^*$.

Remark 4.1 *The state constraint implies the inequality $p(\alpha(0))\geq 0$, which imposes a constraint on $\frac{\partial v}{\partial x}(t,0)$ (Neuman condition) (see Soner (1986a)). In our case, at $x=0$, we have $\alpha(0)=0$ (see Remark 2.2), which imposes a condition on $v(t,0)$(Dirichlet condition).*

The following theorem relates the value function defined in (2.7) with the Bellman equation (4.1).

Theorem 4.1 *Suppose that the value function v belongs to $C^0([0,T]\times I\!R_+)$. Then v is a constrained viscosity solution of (4.1) in $[0,T]\times I\!R_+$.*

Proof. We first prove that v is a supersolution of (4.1) in $[0,T] \times I\!R_+$. Let $(t,x) \in [0,T] \times I\!R_+$ and $\psi \in C^1([0,T] \times I\!R_+)$ such that without loss of generality

$$0 = (v - \psi)(t,x) = \min_{[0,T]\times I\!R_+} (v - \psi).$$

For all $0 < h < T - t$, the dynamic programming principle

$$v(t,x) = \sup_{\alpha_. \in A(t,x)} E\left[v(t+h, X^{t,x,\alpha}_{t+h})\right]$$

implies

$$\psi(t,x) \geq \sup_{\alpha_. \in A(t,x)} E\left[\psi(t+h, X^{t,x,\alpha}_{t+h})\right].$$

Applying Itô's formula to $\psi(t+h, X^{t,x,\alpha}_{t+h})$, we get

$$\sup_{\alpha_. \in A(t,x)} \left\{ \frac{1}{h} E\left[\int_0^h \frac{\partial \psi}{\partial t}(t,x) + A^{\alpha_s}(t,x,\psi(t,x), \frac{\partial \psi}{\partial x}(t,x)) ds\right]\right\} \leq \epsilon(h).$$

Choosing $\alpha_s = \alpha \in A(t,x)$ for all $s \in (0,h)$ and letting $h \to 0^+$, we obtain

$$\frac{\partial \psi}{\partial t}(t,x) + A^{\alpha}\left(t,x,\psi(t,x), \frac{\partial \psi}{\partial x}(t,x)\right) \leq 0.$$

Since

$$\int_t^{t+h} \int_B z \wedge \alpha \, \mu(dt,dz) \longrightarrow \begin{cases} U \wedge \alpha & \text{if there is a claim with size U} \\ 0 & \text{if not,} \end{cases}$$

when $h \to 0^+$, $X^{t,x,\alpha}_{t+h} \geq 0$ P.p.s implies that $x - z \wedge \alpha \geq 0$ for all $z \in B$ and thus

$$\sup_{\alpha \in [0,K] \; x \geq z \wedge \alpha \forall z \in B} \left\{ \frac{\partial \psi}{\partial t}(t,x) + A^{\alpha}(t,x,\psi(t,x), \frac{\partial \psi}{\partial x}(t,x)\right\} \leq 0,$$

which provides the supersolution inequality. It remains to prove the subsolution property. Let $(t,x) \in [0,T] \times I\!R_+$ and $\psi \in C^1([0,T] \times I\!R_+^*)$ such that without loss of generality

$$0 = (v - \psi)(t,x) = \max_{[0,T]\times I\!R_+} (v - \psi).$$

Applying the dynamic programming principle with $0 < h < T - t$, we get

$$v(t,x) = \sup_{\alpha_. \in A(t,x)} E\left[v(t+h \wedge T_1, X^{t,x,\alpha}_{t+h\wedge T_1})\right],$$

where T_1 is the first time jump. This implies

$$\psi(t,x) \leq \sup_{\alpha_. \in A(t,x)} E\left[\psi(t+h\wedge T_1, X^{t,x,\alpha}_{t+h\wedge T_1})\right].$$

Applying Itô's formula to $\psi(t+h\wedge T_1, X^{t,x,\alpha}_{t+h\wedge T_1})$, we obtain

$$\sup_{\alpha_. \in A(t,x)} E\left[\int_0^{h\wedge T_1} \frac{\partial\psi}{\partial t}(t+s, X^{t,x,\alpha}_{t+s}) + A^{\alpha_s}(t+s, X^{t,x,\alpha}_{t+s}, \psi(t+s, X^{t,x,\alpha}_{t+s}), \frac{\partial\psi}{\partial x}(t+s, X^{t,x,\alpha}_{t+s}))ds\right] \geq 0,$$

which implies

$$\begin{aligned}
\epsilon(h) &\leq \sup_{\alpha_. \in A(t,x)} \left\{\frac{1}{h}E\left[\int_0^{h\wedge T_1} \frac{\partial\psi}{\partial t}(t,x) + A^{\alpha_s}(t,x,\psi(t,x),\frac{\partial\psi}{\partial x}(t,x))ds\right]\right\} \\
&\leq E\left[\frac{h\wedge T_1}{h}\right] \sup_{\alpha\in[0,K],\ x\geq z\wedge\alpha\,\forall z\in B} \left\{\frac{\partial\psi}{\partial t}(t,x) + A^{\alpha}(t,x,\psi(t,x),\frac{\partial\psi}{\partial x}(t,x))\right\}. \qquad (4.2)
\end{aligned}$$

Consider the event $\{T_1 < h\}$,

$$\begin{aligned}
P(T_1 < h) &= P(\int_t^{t+h}\int_B z\,\mu(ds,dz) > b) \\
&= P(\int_0^{h}\int_B z\,\mu(ds,dz) > b),
\end{aligned}$$

where the last equality follows from the homogeneity of the integer random valued measure $\mu(dt,dz)$. Since $\mu(dt,dz)$ is cadlag and $\mu(0,B) = 0$, $\int_0^h\int_B z\,\mu(ds,dz) \longrightarrow 0$ when $h \longrightarrow 0^+$. Using the fact that a.s convergence implies convergence in probability, we deduce that $P(\int_0^h\int_B z\,\mu(ds,dz) > b) \longrightarrow 0$ when $h \longrightarrow 0^+$. Taking the limit when $h \rightarrow 0^+$ in Inequality (4.2), we get

$$\sup_{\alpha\in[0,K]\ x\geq z\wedge\alpha\,\forall z\in B} E\left[\frac{\partial\psi}{\partial t}(t,x) + A^{\alpha}(t,x,\psi(t,x),\frac{\partial\psi}{\partial x}(t,x))\right] \geq 0,$$

which proves the subsolution inequality. □

5 Uniqueness

Uniqueness proofs of viscosity solutions of first order integrodifferential operators have been given in Soner (1986b) and Sayah (1991). Since the utility function is not bounded, we shall work in the space of functions $UC_x([0,T]\times \mathbb{R}_+)$.

Sayah (1991) proved a uniqueness result for unbounded viscosity solutions of first order integrodifferential operators. She studied the case where the measure $\pi(dz)$ depends on the state variable x. In our case, the Levy measure is independent of x. We adapt the proof of Pham (1998) to first order integrodifferential operators. We first need an equivalent formulation of viscosity solutions in $C_1([0,T] \times I\!R_+)$ where

$$C_1([0,T] \times I\!R_+) = \{\Phi \in C^0([0,T] \times I\!R_+)\,,\ \sup_{[0,T]\times I\!R_+} \frac{\Phi(t,x)}{1+\mid x \mid} < +\infty\}.$$

(recall that $UC_x([0,T] \times I\!R_+) \subset C_1([0,T] \times I\!R_+)$).

Proposition 5.1 *Let $v \in C_1([0,T] \times I\!R_+)$. Then v is a viscosity supersolution (resp subsolution) of (4.1) in $[0,T] \times I\!R_+$ (resp $[0,T] \times I\!R_+^*$) if and only if*

$$\max_{\alpha\in[0,K]\ x\geq z\wedge\alpha\ \forall z\in B} \left\{ \frac{\partial\psi}{\partial t}(t,x) + A^\alpha(t,x,v(t,x),\frac{\partial\psi}{\partial x}(t,x)) \right\} \geq 0$$

(resp ≤ 0) whenever $\psi \in C^1([0,T] \times I\!R_+)$ and $v - \psi$ has a global minimum (resp maximum) at $(t,x) \in [0,T] \times I\!R_+)$ (resp $(t,x) \in [0,T] \times I\!R_+^*$) .

Proof. The proof is an adaptation of a similar lemma in Soner (1986b). We prove the statement for subsolutions only, the other statement is proved similarly.
Necessary condition: Let $v \in C_1([0,T] \times I\!R_+)$ be a viscosity subsolution, $(t_0,x_0) \in [0,T] \times I\!R_+$ and $\psi\ [0,T] \times I\!R_+ \to I\!R$ such that

$$(v-\psi)(t_0,x_0) = \max_{[0,T]\times I\!R_+} (v-\psi)(t,x).$$

Since $v(t,x) - v(t_0,x_0) \leq \psi(t,x) - \psi(t_0,x_0)$ for all $(t,x) \in [0,T] \times I\!R_+$ and $\pi(dz) \geq 0$ we have

$$A^\alpha(t_0,x_0,v(t_0,x_0),\frac{\partial\psi}{\partial x}(t_0,x_0)) \leq A^\alpha(t_0,x_0,\psi(t_0,x_0),\frac{\partial\psi}{\partial x}(t_0,x_0)).$$

Hence the viscosity property of v provides the expected result.
Sufficient condition: Let $\psi \in C^1([0,T] \times I\!R_+)$ and $(t_0,x_0) \in [0,T] \times I\!R_+$ be such that

$$(v-\psi)(t_0,x_0) = \max_{[0,T]\times I\!R_+} (v-\psi)(t,x) = 0.$$

For each $\epsilon > 0$, we define

$$\Phi^\epsilon(t,x) = \psi(t,x)\chi^\epsilon(t,x) + v(t,x)(1-\chi^\epsilon(t,x)),$$

where χ^ϵ is a smooth function satisfying

$$\begin{aligned} \chi^\epsilon(t,x) &\in [0,1], \\ \chi^\epsilon(t,x) &= 1 \ \text{ if } (t,x) \in B((t_0,x_0),\epsilon), \\ \chi^\epsilon(t,x) &= 0 \ \text{ if } (t,x) \in [0,T] \times I\!R_+ - B((t_0,x_0),2\epsilon). \end{aligned}$$

We have $v(t_0, x_0) = \Phi^\epsilon(t_0, x_0)$ and $v(t,x) - \Phi^\epsilon(t,x) = (v(t,x) - \Phi^\epsilon(t,x))\chi^\epsilon(t,x) \leq 0$.
Hence

$$(v - \Phi^\epsilon)(t_0, x_0) = \max_{(t,x)\in[0,T]\times I\!R_+} (v - \Phi^\epsilon)(t,x).$$

Since v satisfies

$$\max_{\alpha\in[0,K],\ x_0\geq z\wedge\alpha\ \forall z\in B} \left\{ \frac{\partial\psi}{\partial t}(t_0,x_0) + A^\alpha(t_0, x_0, v(t_0,x_0), \frac{\partial\psi}{\partial x}(t_0,x_0)) \right\} \geq 0,$$

and $\nabla\Phi^\epsilon(t_0,x_0) = \nabla\psi(t_0,x_0)$,we obtain:

$$\max_{\alpha\in[0,K],\ x_0\geq z\wedge\alpha\ \forall z\in B} \left\{ \frac{\partial\Phi^\epsilon}{\partial t}(t_0,x_0) + A^\alpha(t_0, x_0, v(t_0,x_0), \frac{\partial\Phi^\epsilon}{\partial x}(t_0,x_0)) \right\} \geq 0.$$

Since $\pi(dz) \geq 0$ and $\Phi^\epsilon(t_0,x_0) = v(t_0,x_0)$,we obtain:

$$\begin{aligned}
&\Big| \max_{\alpha\in[0,K],\ x_0\geq z\wedge\alpha\ \forall z\in B} \left\{ \frac{\partial\Phi^\epsilon}{\partial t}(t_0,x_0) + A^\alpha(t_0, x_0, \Phi^\epsilon(t_0,x_0), \frac{\partial\Phi^\epsilon}{\partial x}(t_0,x_0)) \right\} - \\
&\qquad \max_{\alpha\in[0,K],\ x_0\geq z\wedge\alpha\ \forall z\in B} \left\{ \frac{\partial\Phi^\epsilon}{\partial t}(t_0,x_0) + A^\alpha(t_0, x_0, v(t_0,x_0), \frac{\partial\Phi^\epsilon}{\partial x}(t_0,x_0)) \right\}\Big| \\
&\leq \Big| \max_{\alpha\in[0,K],\ x_0\geq z\wedge\alpha\ \forall z\in B} \left\{ \int_C v(t_0, x_0 - z\wedge\alpha) - \Phi^\epsilon(t_0, x_0 - z\wedge\alpha)\pi(dz) \right\}\Big|.
\end{aligned} \tag{5.1}$$

Observe that, for $(0, -z\wedge\alpha) \notin B((0,0),2\epsilon)$, we have $\Phi^\epsilon(t_0, x_0 - z\wedge\alpha) = v(t_0, x_0 - z\wedge\alpha)$. For $(0, -z\wedge\alpha) \in B((0,0),2\epsilon)$, we have

$$\begin{aligned}
& \mid \Phi^\epsilon(t_0, x_0 - z\wedge\alpha) - v(t_0, x_0 - z\wedge\alpha) \mid \\
= & \mid \psi(t_0, x_0 - z\wedge\alpha) - v(t_0, x_0 - z\wedge\alpha) \mid \chi^\epsilon(t_0, x_0 - z\wedge\alpha) \\
\leq & \mid \psi(t_0, x_0 - z\wedge\alpha) - \psi(t_0,x_0) \mid + \mid v(t_0, x_0 - z\wedge\alpha) - v(t_0,x_0) \mid \\
\leq & \sup_{(t_0,y)\in B((t_0,x_0),2\epsilon)} \mid \frac{\partial\psi}{\partial x}(t_0,y) \mid\mid z\wedge\alpha \mid + w_v(\mid z\wedge\alpha \mid) \\
\leq & \ 2\epsilon \sup_{(t_0,y)\in B((t_0,x_0),2\epsilon)} \mid \frac{\partial\psi}{\partial x}(t_0,y) \mid + w_v(2\epsilon),
\end{aligned} \tag{5.2}$$

where w_v is the modulus of continuity of v. From (5.1) and (5.2), we conclude that v is viscosity subsolution. □

Uniqueness of the solution of the HJB equation (4.1) with terminal condition (2.8) is a consequence of Remark (2.2) and the following theorem:

Theorem 5.1 *Let u (resp v)$\in UC_x([0,T]\times I\!R_+)$ be a viscosity supersolution (resp subsolution) of (4.1) in $[0,T]\times I\!R_+$ (resp $[0,T]\times I\!R_+^*$), then*

$$u(t,x)-v(t,x)\leq \sup_{(t,x)\in\partial Q}\{u(t,x)-v(t,x)\}, \tag{5.3}$$

with $\partial Q=[0,T[\times\{0\}\bigcup\{T\}\times I\!R_+$.

Proof. It suffices to prove Inequality (5.3) for $t>0$. By continuity of u and v, Inequality (5.3) will then be satisfied for all $t\in[0,T]$ and $x\geq 0$.
For β, ϵ, δ, $\lambda>0$, define Φ in $]0,T]\times I\!R_+\times I\!R_+$ as

$$\Phi(t,x,y):=u(t,x)-v(t,y)-\frac{\beta}{t}-\frac{1}{2\epsilon}(x-y)^2-\delta\exp\left(\lambda(T-t)\right)(x^2+y^2).$$

Since $u,v\in C_1([0,T]\times I\!R_+)$, there exists $(t^*,x^*,y^*)\in[0,T]\times I\!R_+\times I\!R_+$ which maximizes Φ. By using $2\Phi(t^*,x^*,y^*)\geq\Phi(t^*,x^*,x^*)+\Phi(t^*,y^*,y^*)$ and the uniform continuity of u and v, we easily check (see Ishii H. (1984)) that:

$$\frac{1}{\epsilon}(x^*-y^*)^2\leq w(c\sqrt{\epsilon}), \tag{5.4}$$

where c is independent of β, ϵ, δ, λ and w is the modulus of continuity of u and v.
>From the inequality $\Phi(T,0,0)\geq\Phi(t^*,x^*,y^*)$, and since $u,v\in C_1([0,T]\times I\!R_+)$, we obtain that

$$\delta(x^{*2}+y^{*2})\leq c(1+x^*+y^*).$$

Using Young's inequality, we deduce that

$$\mid x^*\mid,\mid y^*\mid\leq C_\delta, \tag{5.5}$$

where C_δ is a constant depending on δ but not on ϵ, and

$$\delta x^*\,,\,\delta y^*\ \text{ is bounded uniformly in } \delta. \tag{5.6}$$

Using (5.4) and (5.5) along a subsequence, (t^*,x^*,y^*) converges when $\epsilon\to 0$. Let's denote $(\bar{t},\bar{x},\bar{x})$ its limit.
If $t^*=T$, then from the inequality $\Phi(t,x,x)\leq\Phi(T,x^*,y^*)$ we deduce that:

$$\begin{aligned}
& u(t,x)-v(t,x)-\frac{\beta}{t}-2\delta\exp\left(\lambda(T-t)\right)x^2\\
\leq\ & u(T,x^*)-v(T,x^*)-v(T,y^*)+v(T,x^*)\\
\leq\ & \sup_{(t,x)\in\partial Q}\{u(t,x)-v(t,x)\}+w(\mid x^*-y^*\mid).
\end{aligned}$$

Sending $\beta, \delta, \epsilon \to 0^+$, we have

$$u(t,x) - v(t,x) \leq \sup_{(t,x)\in\partial Q} \{u(t,x) - v(t,x)\}. \tag{5.7}$$

Assume that $t^* < T$.
If $x^* = 0$, then

$$\begin{aligned}
& u(t,x) - v(t,x) - \frac{\beta}{t} - 2\delta \exp\left(\lambda(T-t)\right)x^2 \\
\leq\ & u(t^*,0) - v(t^*,0) + v(t^*,0) - v(t^*,y^*) \\
\leq\ & \sup_{(t,x)\in\partial Q} \{u(t,x) - v(t,x)\} + w(|\, y^* \,|).
\end{aligned}$$

Let $\beta, \delta, \epsilon \to 0^+$. We obtain

$$u(t,x) - v(t,x) \leq \sup_{(t,x)\in\partial Q} \{u(t,x) - v(t,x)\}. \tag{5.8}$$

When $y^* = 0$, we conclude similarly.
When x^* and $y^* \neq 0$, applying the lemma 2 of Crandall and Lions (1986), we obtain

$$\begin{aligned}
& \frac{\beta}{t^2} + \delta\lambda \exp\left(\lambda(T-t^*)\right)(x^{*2} + y^{*2}) \\
\leq\ & \max_{\alpha\in[0,A],\, x^*\geq z\wedge\alpha} \left\{ p(\alpha)(\frac{2}{\epsilon}(x^* - y^*) + 2\delta \exp(\lambda(T-t^*)x^*) + B^\alpha(t^*, x^*, u(t^*,x^*)) \right\} \\
-\ & \max_{\alpha\in[0,A],\, y^*\geq z\wedge\alpha} \left\{ p(\alpha)(\frac{2}{\epsilon}(x^* - y^*) - 2\delta \exp(\lambda(T-t^*))y^*) + B^\alpha(t^*, y^*, v(t^*,y^*)) \right\} \\
\leq\ & \max_{\alpha\in[0,A],\, x^*\geq z\wedge\alpha,\, y^*\geq z\wedge\alpha} \left\{ 2p(\alpha)\delta \exp(\lambda(T-t^*))(x^* + y^*) \right\} \\
+\ & \max_{\alpha\in[0,A],\, x^*\geq z\wedge\alpha,\, y^*\geq z\wedge\alpha} \left\{ B^\alpha(t^*, x^*, u(t^*,x^*)) - B^\alpha(t^*, y^*, v(t^*,y^*)) \right\} \\
:=\ & I_1 + I_2,
\end{aligned}$$

where $B^\alpha(t,x,w(t,x)) := \int_B w(t, x - z\wedge\alpha) - w(t,x)\pi(dz)$. Since $p(\alpha)$ is bounded uniformly in α and by using (5.6), we have $I_1 \leq C \exp(\lambda(T-t^*))$.
In order to get an upper bound of the term I_2, we need to evaluate the following term

$$\begin{aligned}
& u(t^*, x^* - z\wedge\alpha) - u(t^*,x^*) - v(t^*, y^* - z\wedge\alpha) + v(t^*,y^*) \\
=\ & \Phi(t^*, x^* - z\wedge\alpha, y^* - z\wedge\alpha) - \Phi(t^*, x^*, y^*) \\
+\ & \delta \exp(\lambda(T-t^*))((x^* - z\wedge\alpha)^2 + (y^* - z\wedge\alpha)^2 - x^{*2} - y^{*2}) \\
\leq\ & \delta \exp(\lambda(T-t^*))(x^{*2} + y^{*2}),
\end{aligned}$$

and so $I_2 \leq \delta \exp(\lambda(T-t^*))(x^{*2}+y^{*2}) \int_B m(dz)$. Sending $\epsilon \to 0$, we obtain

$$\frac{\beta}{t^2} \leq \exp(\lambda(T-t^*))(C + 2\delta C \bar{x}^2 - 2\delta\lambda\bar{x}^2).$$

For λ large enough, we obtain $\beta \leq 0$ which is impossible, so $x^* = 0$ or $y^* = 0$, which proves the comparison theorem (5.1) . □

6 Numerical study

Here we restrict ourselves to the case where the integer valued random measure $\mu(dt, dz)$ is a Poisson process with constant intensity π. All the claims have the same size denoted by δ. Since the amounts are relatives, we take $\delta = 1$. We consider the HARA utility function $U(x) = \dfrac{x^\gamma}{\gamma}$ with $\gamma = 0.5$. We choose $K = \delta$. Our purpose is to solve the following equation

$$\begin{cases} \max\limits_{\alpha\in[0,\delta]\ x\geq\alpha} \left\{ \dfrac{\partial v}{\partial t}(t,x) + A^\alpha(t,x,v(t,x),\dfrac{\partial v}{\partial x}(t,x)) \right\} = 0, \quad \text{for all } (t,x) \in [0,T[\times \mathbb{R}_+^* \\ v(t,0) = 0, \quad \text{for all } t \in [0,T], \\ v(T,x) = U(x), \quad \text{for all } x \in \mathbb{R}_+, \end{cases} \tag{6.1}$$

where

$$A^\alpha(t,x,v(t,x),\frac{\partial v}{\partial x}(t,x)) = p(\alpha)\frac{\partial v}{\partial x}(t,x)) + \pi(v(t,x-\alpha) - v(t,x)).$$

We proceed with a technical change of variable which brings $[0,T]\times\mathbb{R}_+$ into $[0,T]\times[0,1]$, namely

$$\begin{cases} z = \frac{x}{1+x} \\ \psi(t,z) = (1-z)v(t,x). \end{cases}$$

The function ψ is defined in $[0,T]\times[0,1]$ and satisties

$$\begin{cases} \max\limits_{\alpha\in[0,\delta]\ \frac{z}{1-z}\geq\alpha} \{ \dfrac{\partial \psi}{\partial t}(t,z) + \bar{A}^\alpha(t,z,\psi(t,z),\dfrac{\partial \psi}{\partial z}(t,z))\} = 0, \quad \text{for all } (t,z) \in [0,T]\times(0,1) \\ \psi(t,0) = 0, \quad \text{for all } t \in [0,T] \\ \psi(t,1) = 0, \quad \text{for all } t \in [0,T] \\ \psi(T,z) = U(\frac{z}{1-z})(1-z), \quad \text{for all } z \in (0,1), \end{cases} \tag{6.2}$$

where

$$\begin{aligned} \bar{A}^\alpha(t,z,\psi(t,z),\frac{\partial \psi}{\partial z}(t,z)) &= p(\alpha)(1-z)^2\frac{\partial \psi}{\partial z}(t,z)) + p(\alpha)(1-z)^2\psi(t,z)) \\ &+ \pi((1-(1-z)\alpha)\psi(t,\frac{z-(1-z)\alpha}{1-(1-z)\alpha}) - \psi(t,z)). \end{aligned}$$

In Section 3, we have proved that the value function (2.7), within a change of variables, is the unique viscosity solution of Equation (6.2). This solution can be approximated by the following numerical method:
(i) approximate Equation (6.2) by using a consistant finite difference approximation which satisfies the discrete maximum principle (DMP) (see Lapeyre, Sulem and Talay),
(ii) solve the discrete equation by means of the Howard algorithm (policy iteration) (see Howard (1960)). Finally a reverse change of variables is performed in order to display results of Equation (6.1).

6.1 Finite Difference Approximation

Let $h = \frac{T}{N}$, $(N \in \mathbb{N}^*)$ denote the finite difference step in the time coordinate, $p = \frac{1}{M}$, $(M \in \mathbb{N}^*)$ denote the finite difference step in the state coordinate. Let $(t_i, z_j) = (ih, jp)$ denote the points of the grid $\Omega_{p,h} = [0,T] \times (0,1) \cap (h\mathbb{Z} \times p\mathbb{Z})$, $0 \le i \le N-1$, $0 \le j \le M-1$. Equation (6.2) is discretized by replacing the first partial derivatives of ψ by the following approximation

$$\frac{\partial \psi}{\partial t}(t_i, z_j) \simeq \frac{\psi((i+1)h, jp) - \psi(ih, jp)}{h},$$

$$\frac{\partial \psi}{\partial z}(t_i, z_j) \quad \simeq \quad \begin{cases} \frac{\psi(ih,(j+1)p) - \psi(ih,jp)}{p}, & if\ p(\alpha) \ge 0 \\ \frac{\psi(ih,jp) - \psi(ih,(j-1)p)}{p}, & if\ p(\alpha) < 0. \end{cases}$$

For the boundary conditions, we set

$$\begin{aligned} \psi(t_i, 0) &= 0 \ \text{ for all } t_i \in [0,T] \cap h\mathbb{Z} \\ \psi(t_i, 1) &= 0 \ \text{ for all } t_i \in [0,T] \cap h\mathbb{Z} \\ \psi(T, z_j) &= U(\frac{z_j}{1-z_j})(1-z_j) \ \text{ for all } z_j \in (0,1) \cap p\mathbb{Z}. \end{aligned}$$

We choose a fully implicit θ-scheme, which leads to a system of $N \times (M-1)$ equations with $N \times (M-1)$ unknowns $\{\psi(ih, jp)\ , (ih, jp) \in \Omega_{p,h}\}$:

$$\begin{cases} \max_{\alpha \in \mathcal{U}} \left\{ h\bar{A}_p^{\alpha,ih} \psi(ih, jp) + \psi((i+1)h, jp) - \psi(ih, jp) \right\} = 0, & \text{for all } (ih, jp) \in \Omega_{p,h}, \\ \psi(T, jp) = U(\frac{jp}{1-jp})(1-jp) \ \text{ for all } jp \in (0,1) \cap p\mathbb{Z} \end{cases} \tag{6.3}$$

where

$$\mathcal{U} = \{(\alpha_{ij})_{0 \le i \le N-1\,,\,1 \le j \le M-1}, \alpha_{ij} \in [0, \delta] \ \text{ and } \ \frac{z_j}{1-z_j} \ge \alpha_{ij}\,,\ 0 \le i \le N-1\,,\ 1 \le j \le M-1\}$$

and $\bar{A}_p^{\alpha,ih}$ is the $(M-1)\times(M-1)$ matrix associated to the approximation of the operator $\bar{A}^\alpha$ at time ih. Let $\mathcal{A}_p$ denote the set of control functions $\alpha\ :\Omega_{p,h}\longrightarrow U$. The system of equations (6.3) can be written as a system of N stationary Bellman equations:

$$\left\{\begin{array}{l}\max\limits_{\alpha\in\mathcal{A}_p}\left\{h\bar{A}_p^{\alpha,ih}\psi_{h,p}^{ih}+\psi_{h,p}^{(i+1)h}-\psi_{h,p}^{ih}\right\}=0,\ i=0...N-1,\\ \psi_{h,p}^{T}=(U(\frac{jp}{1-jp})(1-jp))_{j=1...M-1}\end{array}\right. \tag{6.4}$$

where $\psi_{h,p}^{ih}$ a vector which approximate $(\psi(ih,jp))_{j=1...M-1}$.The system of N stationary Bellman equations (6.4) can be solved by value iteration or Howard algorithms. We describe below these algorithms.

6.2 The value iteration method

Suppose that the matrix $\bar{A}_p^{\alpha,ih}$, $i=0...N-1$, is diagonally dominant, that is, there exists $r>0$ such that

$$(\bar{A}_p^{\alpha,ih})_{j,k}\geq 0\ \forall j\neq k\ \text{ and }\ \sum_{k=1}^{M-1}(\bar{A}_p^{\alpha,ih})_{j,k}=-r<0\ \forall\ 1\leq j\leq M-1, \tag{6.5}$$

then $\bar{A}_p^{\alpha,ih}$, $i=0...N-1$ satisfies the discrete maximum principle (i.e $(\bar{A}_p^{\alpha,ih}\psi_{h,p}^{ih}\leq 0\ )\Longrightarrow(\psi_{h,p}^{ih}\leq 0\ ))$, which is a sufficient condition for the stability of (6.4). In our case Property (6.5) is not verified, however stability of $\bar{A}_p^{\alpha,ih}$, $i=0...N-1$, is confirmed by numerical experiments. Suppose that Property (6.5) is verified, then there exists $k>0$ and a submarkovian matrix $M_p^{\alpha,ih}$, $i=0...N-1$ such that

$$\bar{A}_p^{\alpha,ih}=-rI+\frac{1}{k}(M_p^{\alpha,ih}-I),$$

where I is the identity matrix. Problem (6.4) is then equivalent to

$$\left\{\begin{array}{l}\max\limits_{\alpha\in\mathcal{A}_p}\left\{h(-rI+\frac{1}{k}(M_p^{\alpha,ih}-I))\psi_{h,p}^{ih}+\psi_{h,p}^{(i+1)h}-\psi_{h,p}^{ih}\right\}=0,\ i=0...N-1,\\ \psi_{h,p}^{T}=(U(\frac{jp}{1-jp})(1-jp))_{j=1...M-1}\end{array}\right.$$

which implies that

$$\psi_{h,p}^{ih}=\max_{\alpha\in\mathcal{A}_p}\left\{\frac{\frac{h}{k}}{1+rh+\frac{h}{k}}M_p^{\alpha,ih}\psi_{h,p}^{ih}+\frac{\psi_{h,p}^{(i+1)h}}{1+rh+\frac{h}{k}}\right\}\ i=0...N-1.$$

Since the contracting factor $\frac{\frac{h}{k}}{1+rh+\frac{h}{k}}$ is less than 1 and $\|M_p^\alpha\|\leq 1$, $\psi_{h,p}^{ih}$, $i=0...N-1$ exists and is unique. Under the hypothesis that the discrete maximum principle holds, Equation (6.4) can then be interpreted as a discrete-time HJB equation.

6.3 The Howard algorithm

To solve Equation (6.4), we use the Howard algorithm (see Lapeyre Sulem and Talay), also named policy iteration. It consists on computing two sequences $(\alpha^{ih,n})_{n\in I\!N}$ and $(\psi_{h,p}^{ih,n})_{n\in I\!N}, i = 0...N-1$, (starting from α_0) defined by:

- Step $2n$. To the strategy $\alpha^{ih,n}$, we associate the solution $\psi_{h,p}^{ih,n}$ of

$$(h\bar{A}_{h,p}^{\alpha^{ih,n},ih} - I)\psi_{h,p}^{ih,n} + \psi_{h,p}^{(i+1)h,n} = 0,\ i = 0...N-1.$$

- Step $2n+1$. To $\psi_{h,p}^{ih,n}$ is associated another strategy $\alpha^{ih,n+1}$

$$\alpha^{ih,n+1} \in \arg\max_{\alpha\in\mathcal{A}_p} \left\{(h\bar{A}_{h,p}^{\alpha,ih} - I)\psi_{h,p}^{ih,n} + \psi_{h,p}^{(i+1)h,n}\right\} = 0,\ i = 0...N-1.$$

When $\bar{A}_{h,p}^{\alpha,ih}, i = 0...N-1$ satisfies the discrete maximum principle, the sequence $(\psi_{h,p}^{ih,n})_{n\in I\!N}, i = 0...N-1$ increases and is bounded and so converges to the solution of (6.4).

6.4 Numerical results

Equation (6.1) is solved by using the Howard algorithm. Two tests are performed:
Test1: $\delta = 1 \quad h = p = 0.02 \quad \pi = 5$
Test2: $\delta = 1 \quad h = p = 0.02 \quad \pi = 1$.
For Test1 (resp Test2), the optimal policy, the value function and their contour lines are displayed in figures 1-3 (resp figures 4-6). The Howard algorithm is very efficient: it converges in two iterations. We observe that the compagny of insurance increases its risk when its reserve increases which is coherent with the intuition. When the intensity π increases, the value function v and the optimal policy of reinsurance decreases. The dependence of the value function on time is shown in figures 3 and 6.

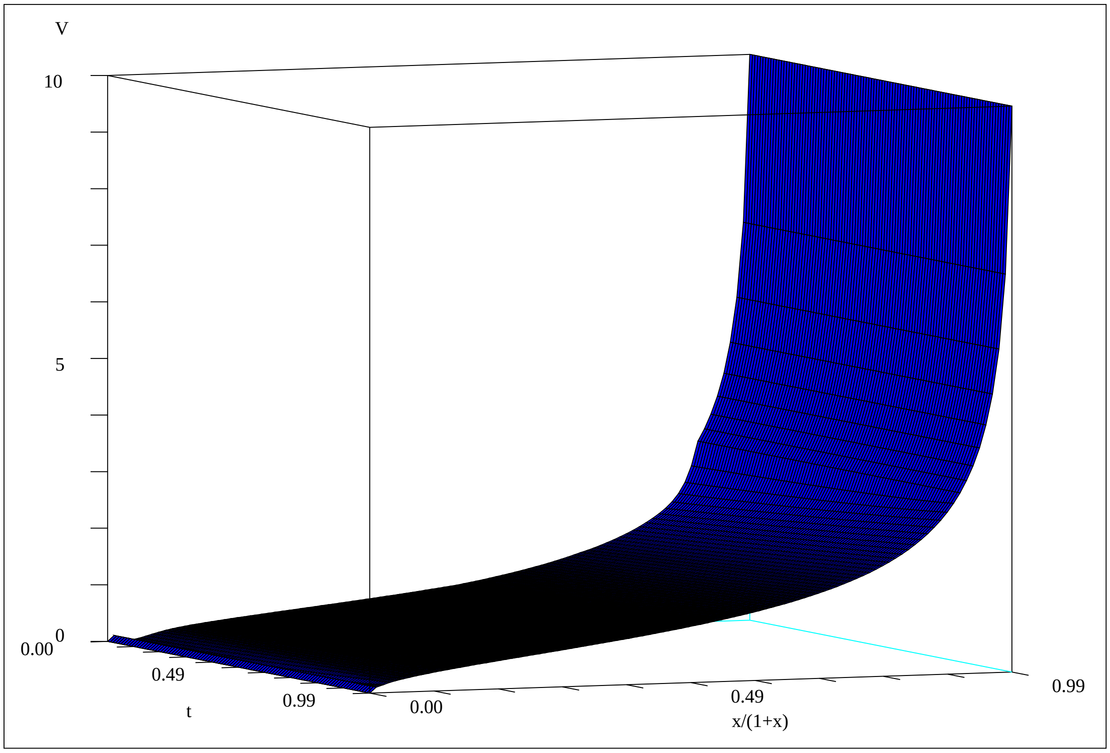

FIG.1. The value function

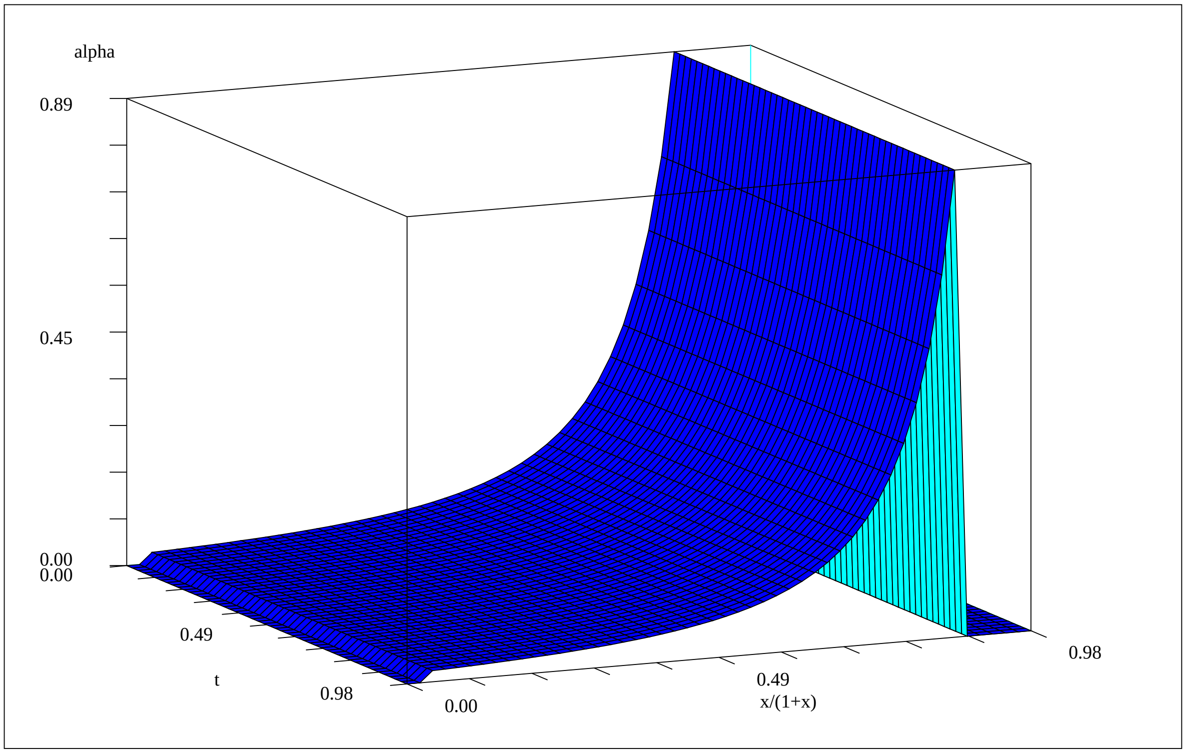

FIG.2. Optimal policy of Reinsurance

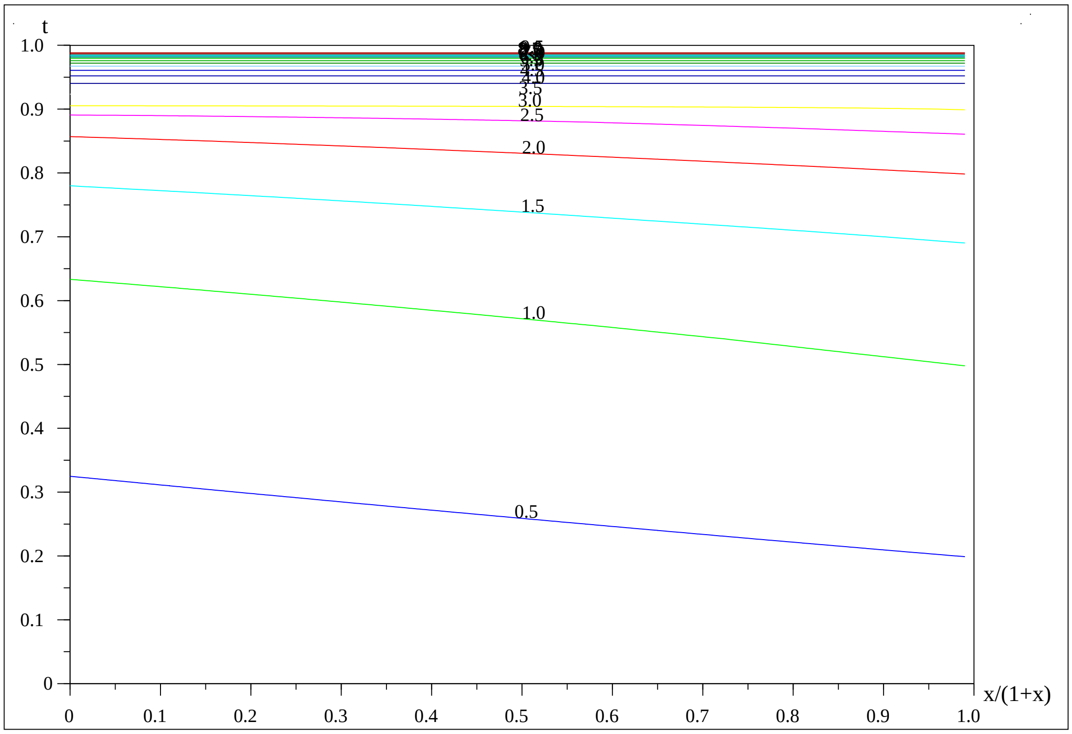

FIG. 3. Contour lines of the value function

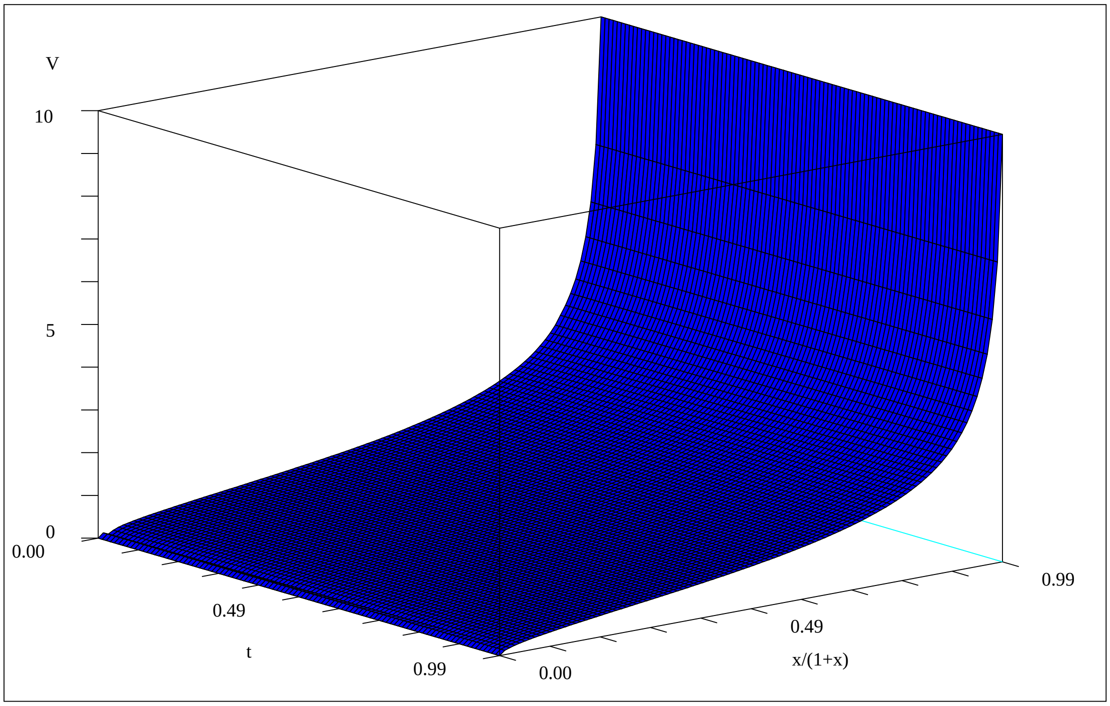

FIG.4. The value function

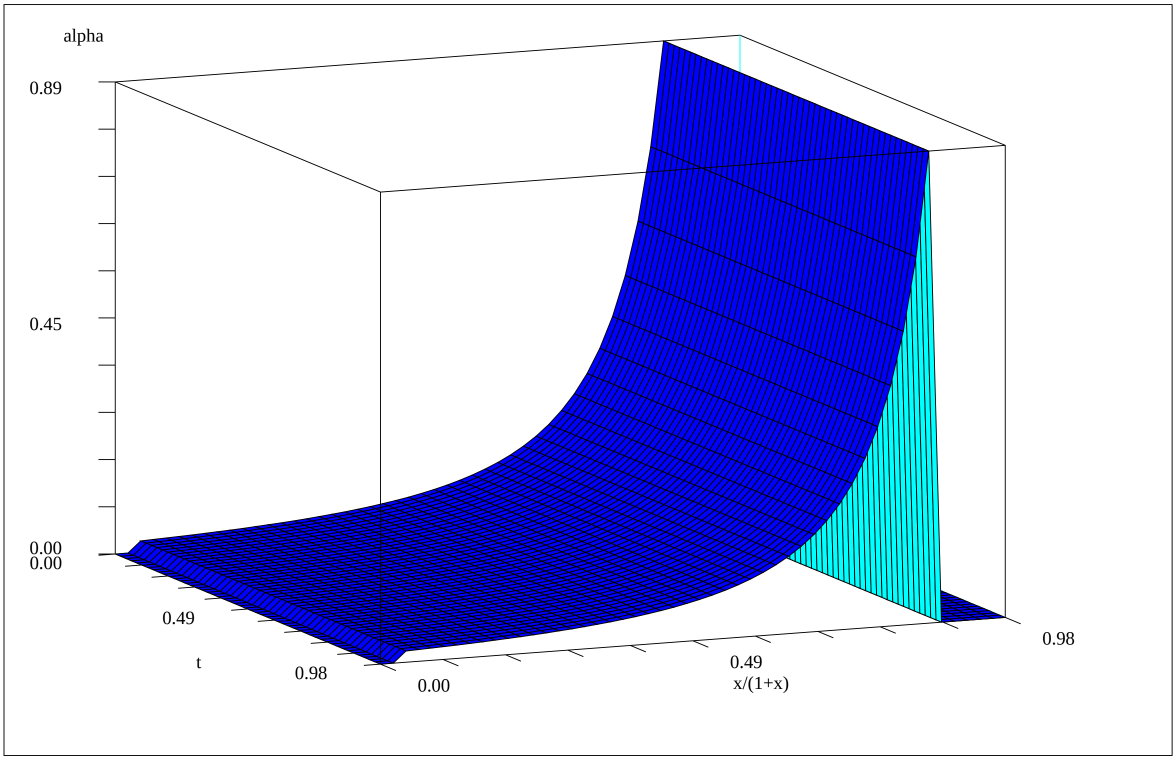

FIG.5. Optimal Policy of Reinsurance

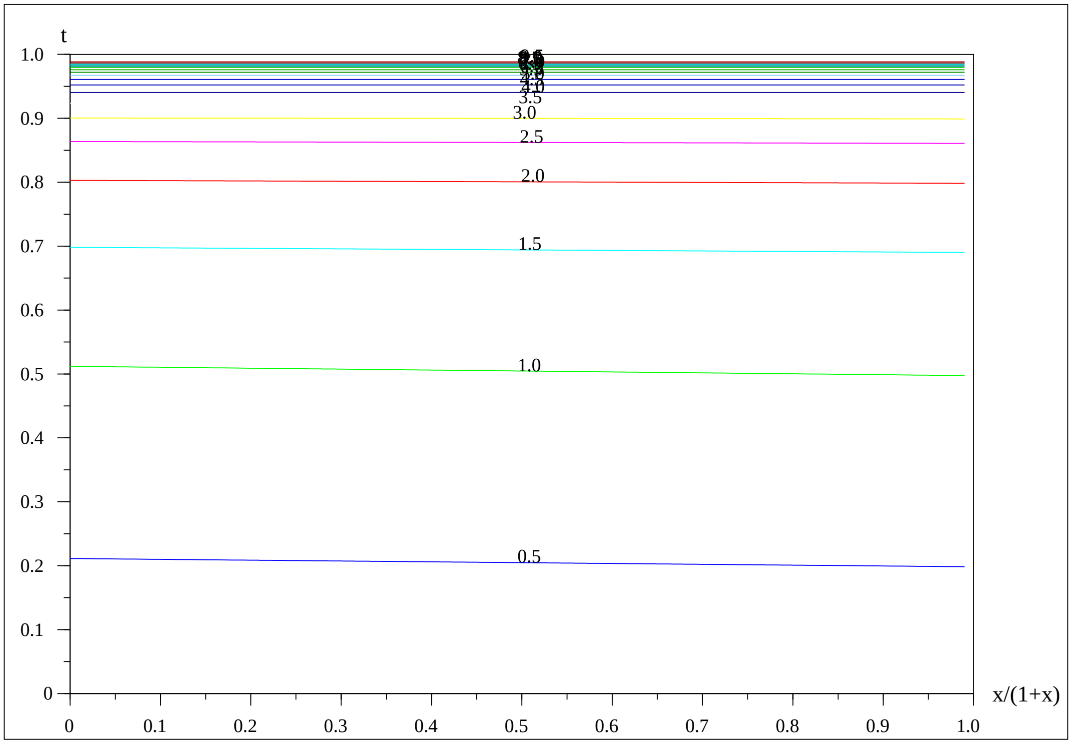

FIG. 6. Contour lines of the value function

Acknowledgments. We are very grateful to Huyên Pham for helpful comments and advices.

References

[1] Akian M., J.L. Menaldi and A. Sulem (1996): "On an investment-consumption mdel with transaction costs",*SIAM J. Control and Optimization*, 34. 329-364.

[2] Asmussen S. Højgaard B. and M. Taksar (2000) : "Optimal risk control and dividend distribution policies. Example of excess-of loss reinsurance for an insurance corporation",*Finance and Stochastics*, 4. 299-324.

[3] Barles G. and P.E. Souganidis (1991): "Convergence of approximation schemes for fully nonlinear second order equations",*Asymptotic Analysis*, 4. 271-283.

[4] Benth F.E. , Karlsen K.H. and K. Reikvam(2001): "Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: A viscosity solution approach",*Finance and Stochastics*,5. 275-303.

[5] Brémaud P. (1981) : Point Processes and Queues, Springer Verlag.

[6] Crandall M.G. and P.L.Lions(1983): "Viscosity solutions of Hamilton-Jacobi equations",*Trans. Amer. Math. Soc*,277. 1-42.

[7] Crandall M.G. and P.L.Lions(1986): "On existence and uniqueness of solutions of Hamilton-Jacobi equations",*Nonlinear Analysis, Theory, Methods and Applications*,4. 353-370

[8] Fleming W.H. and H.M. Soner(1993): Controlled Markov Processes and Viscosity solutions, Springer-Verlag.

[9] Ishii H. (1984), "Uniqueness of Umbounded viscosity solutions of Hamilton-Jacobi Equations",*Ind. Univ. Math. J.*,33, 721-748.

[10] Lapeyre B., Sulem A. and D. Talay: Understanding Numerical Analysis for Option Pricing, editor: L. C.G.Rogers and D. Talay, Cambridge University Press, to appear.

[11] Hipp C. and M. Taksar (1999) : "Stochastic Control for Optimal New Business", preprint.

[12] Howard R. A (1960): Dynamic Programming and Markov Processes, MIT Press, Cambridge, MA.

[13] Mnif M. and H. Pham(2001): "Stochastic optimization under constraints",*Stochastic Processes and their Applications*,93. 149-180.

[14] Pham H. (1998): "Optimal stopping of controlled jump diffusion processes: a viscosity solution approach",*J. Math. Syst. Estim. Control*,8. 27pp.

[15] Sayah A. (1991): "Equation d'Hamilton-Jacobi du premier ordre avec terme integro-differentiel: Parties I et II",*Commun. in Partial Differential Equations*,16. 1057-1074.

[16] Schmidli H. (1999) : "Optimal Proportional Reinsurance Policies in a Dynamic Setting", to appear in *Scandinavian Actuarial Journal*.

[17] Soner H.M. (1986a): "Optimal control with state-space constraint I",*SIAM J. Control and Optimization* ,24. 552-561.

[18] Soner H.M. (1986b): "Optimal control with state-space constraint II",*SIAM J. Control and Optimization* ,24. 1110-1122.

[19] Touzi N. (2000): "Optimal insurance demand under marked point processes shocks",*Annals of Applied Probability*,10. 283-312.

Unité de recherche INRIA Rocquencourt
Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399